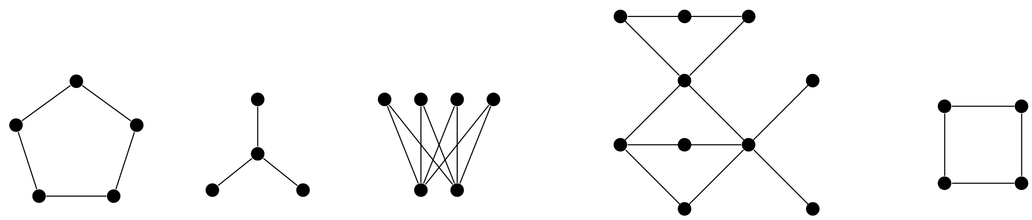
Chapter 2 - Matchings and Coverings - Tutte's Theorem

Let G be a graph. Let C_G the set of its connected components, and by $q(G)$ the number of its connected components of odd order (odd components).

1: Find $S \subseteq V(G)$ such that $q(G-S) > |S|$.



2: Show that if G has a perfect matching (1-factor), then

$$q(G-S) \leq |S| \quad \text{for all } S \subseteq V(G). \qquad \textit{(Tutte's condition)}$$

Solution: Let M be a 1-factor (perfect matching). Let $S \subseteq V$. For each odd component X_i exists $x_i \in X_i$ that is matched by M to a vertex y_i not in X_i. This means $y_i \in S$. Since M is a matching, $y_i \neq y_j$ if $i \neq j$. Hence $|S| \geq q(G-S)$.

Theorem (Tutte 1947)
A graph G has a perfect matching (1-factor) if and only if $q(G-S) \leq |S|$ for all $S \subseteq V(G)$.

Proof: "If G has a 1-factor then it satisfies Tutte's condition." is already proved.

Goal: "If G does not have a 1-factor then G does not satisfy Tutte's condition."
I.e. If no 1-factor, then there is $S \subset V(G)$ such that $q(G-S) > |S|$.

Trick: Let G' be a maximal graph without 1-factor such that G is a subgraph of G'.

3: Show that if $S \subseteq V(G') = V(G)$ and $q(G'-S) > |S|$, then also $q(G-S) > |S|$.

Solution: If vertices belong to different components in $G'-S$, they are also in different components of $G-S$. So components do not merge. Now if X was a component in $G'-S$, it may fall apart into several components. But if $|X|$ is odd, at least one of the components in $G[X]$ is odd. So $q(G-S) \geq q(G'-S)$.

Now we continue with G' only.

4: Show that G' is not a complete graph.

Solution: If G' is a complete graph, then it does not have a 1-factor iff $|V(G)|$ is odd and then $S = \emptyset$ will work.

Our main goal is to show that every component of $G' - S$ is complete. First we show how it is useful.

5: Assume that every component of $G' - S$ is complete, i.e. induces a clique. Now show that if $q(G - S) \leq |S|$, then G' has a perfect matching.

Solution: Now vertices in each even component of $G - S$ can be perfectly matched. In odd component, all but one vertex can be matched. This remaining vertex can be match to S, as long as $q(G - S) \leq |S|$. Finally, if there are any vertices in S still unmatched, they can be matched to each other.

Now towards the complete components. Let X be vertices of a component of $G' - S$ and assume $G'[X]$ is not complete. This gives vertices $a, a' \in X$ such that $aa' \notin E(G')$. Let $a, b, c, \ldots$ be the shortest $a - a'$ path in $G'[X]$. Notice that $ac \notin E(G')$. (Why)? Since $b \notin S$, there exists $d \in V(G')$ such that $bd \notin E(G')$.

By maximality of G', $G' + ac$ contains a 1-factor M_1 and $G' + bd$ contains a 1-factor M_2.

6: Make a sketch of the situation. Now our goal is to find a 1-factor in G', giving a contradiction. We try to get it from M_2 by replacing edge bd. Consider a union of M_1 and M_2. Then bd is in an even cycle C. How to use this even cycle to obtain a 1-factor in G'? What if $ac \in C$?

Solution: First, if $ac \notin C$, then $C - M_2$ are all edges of G. Notice that swapping M_2 for M_1 on C makes a 1-factor that exists in G, done.

Now $ac \in C$. Then we cannot do the switch, but we can still find an even cycle, that allows for the swith. Assume that the cycle C looks like $b, d, \ldots, a, c, \ldots$. Take a cycle $C' = b, d, \ldots, a, b$. Notice it is still an even cycle and not all edges of $C' - M_2$ are in G'. We can then switch on C' instead of C.

Corollary (Petersen 1891) Every bridgeless cubic graph has a 1-factor.

7: Prove the corollary. Here is the plan. Let G be a cubic bridgeless graph. Pick $S \subseteq V(G)$ and verify Tutte's condition. Hint: If $G[X]$ is an odd component, how many edges go between $G[X]$ and S? (use bridgeless and handshake lemma)

Solution: